

# Non-perturbative predictions for cold atom Bose gases with tunable interactions

Fred Cooper,<sup>1,2</sup> Chih-Chun Chien,<sup>1</sup> Bogdan Mihaila,<sup>1</sup> John F. Dawson,<sup>3</sup> and Eddy Timmermans<sup>1</sup>

<sup>1</sup>*Los Alamos National Laboratory, Los Alamos, NM 87545*

<sup>2</sup>*Santa Fe Institute, Santa Fe, NM 87501*

<sup>3</sup>*Department of Physics, University of New Hampshire, Durham, NH 03824*

We derive a theoretical description for dilute Bose gases as a loop expansion in terms of composite-field propagators by rewriting the Lagrangian in terms of auxiliary fields related to the normal and anomalous densities. We demonstrate that already in leading order this non-perturbative approach describes a large interval of coupling-constant values, satisfies Goldstone's theorem, yields a Bose-Einstein transition that is second-order, and is consistent with the critical temperature predicted in the weak-coupling limit by the next-to-leading order large- $N$  expansion.

PACS numbers: 03.75.Hh, 05.30.Jp, 67.85.Bc

Nearly a century after the first observation of the lambda transition in liquid helium[1], a quantitative, first-principles description of strongly-correlated bosons remains a challenge. After the transition was recognized as the onset of superfluidity[2], the connection with Bose-Einstein condensation (BEC) was proposed[3], but it was Bogoliubov's work[4] pointing out that the dispersion of the elementary BEC excitations satisfy the Landau criterion for superfluidity[5] that motivated weakly-interacting BEC studies to investigate superfluid properties. In weakly-interacting systems, the many-body properties do not depend on the shape of the interaction potential, but only on the  $s$ -wave scattering length,  $a_0$ , and the boson fluid acts as point-like interacting particles[6].

Unlike liquid helium, cold atoms remain point-like even when the scattering length is tuned near a Feshbach resonance. Then, strongly-correlated cold atom bosons offer the exciting prospect of studying point-like strongly interacting bosons, possibly in the universal regime where the scattering length greatly exceeds the inter-particle distance and the latter becomes the only relevant length scale[7]. This hope appeared thwarted when it was shown that the three-body loss rate in cold atom traps scales as  $a_0^4$  near a Feshbach resonance[8]. In accordance, the universal regime was reached only in ultra-cold fermionic gases[9], where the three-body loss is reduced by virtue of the Pauli exclusion principle. However, the recent observation that three-body losses are strongly suppressed in optical lattices when the average number of bosons per site is two or less[10], rekindles the prospect of studying medium and strongly-correlated cold atom bosons. Novel cold-atom trap technologies that produce stable, flat potentials bound by a sharp edge[11], suggest the study of finite-temperature properties such as the BEC transition temperature  $T_c$  and the superfluid to normal fluid ratio and depletion, at fixed density,  $\rho$ .

At finite temperature, the description of BEC's remains a challenge even in the weakly-interacting regime. Standard approximations such as the Hartree-Fock-Bogoliubov and the Popov schemes, generally fall within the Hohenberg and Martin classification[12] of conserv-

ing and gapless approximations, which implies that they either violate Goldstone's theorem or general conservation laws[13]. These approximations generally predict the BEC transition to be a first-order transition, whereas we expect the transition to be second order[14].

In this paper, we present a new theoretical framework that describes a large interval of  $\rho^{1/3}a_0$ -values, satisfies Goldstone's theorem and yields a Bose-Einstein transition that is second-order, while also predicting reasonable values for the depletion. Furthermore, this framework can predict *all* experimentally relevant quantities within the same calculation, determining fully consistently quantities such as  $T_c$ , the collective mode frequencies[15] and the compressibility (which characterizes the density profile in a shallow trap[16]). In contrast with other resummation schemes, such as the large- $N$  expansion[17] or functional renormalization techniques[18], here we treat the normal and anomalous densities on equal footing.

In our approach, we generate a one-parameter family of equivalent Lagrangians. We choose this parameter to reproduce the one-loop result at mean-field level in the weakly-interacting limit. Thus, we identify the optimal auxiliary-field Lagrangian for the purpose of a systematic non-perturbative expansion. Then, the critical temperature variation in leading order is the same as the one found in the next-to-leading order large- $N$  expansion.

In dilute bosonic gas systems, the classical action is given by  $S[\phi, \phi^*] = \int dx \mathcal{L}[\phi, \phi^*]$ , with  $dx \equiv dt d^3x$  and the Lagrangian density

$$\mathcal{L}[\phi, \phi^*] = \frac{i\hbar}{2} [\phi^*(x) (\partial_t \phi(x)) - (\partial_t \phi^*(x)) \phi(x)] - \phi^*(x) \left\{ -\frac{\hbar^2 \nabla^2}{2m} - \mu \right\} \phi(x) - \frac{\lambda}{2} |\phi(x)|^4. \quad (1)$$

Here,  $\mu$  is the chemical potential and the coupling is  $\lambda = 4\pi\hbar^2 a_0/m$ . To account for the contributions of the normal and anomalous densities, we use the Hubbard-Stratonovitch transformation[19] to introduce the real and complex auxiliary fields (AF),  $\chi(x)$  and  $A(x)$ . We

add to Eq. (1) the AF Lagrangian density[20, 21]

$$\mathcal{L}_{\text{aux}}[\phi, \phi^*, \chi, A, A^*] = \frac{1}{2\lambda} [\chi(x) - \lambda \cosh \theta |\phi(x)|^2]^2 - \frac{1}{2\lambda} |A(x) - \lambda \sinh \theta \phi^2(x)|^2, \quad (2)$$

where  $\theta$  is the mixing parameter between the normal and anomalous densities,  $\chi(x)$  and  $A(x)$ . The usual large-N approximation[21] is obtained when  $\theta = 0$ . Then, the action becomes

$$\begin{aligned} S[\Phi, J] &= S[\phi_a, \chi, A, A^*, j_a, s, S] \\ &= -\frac{1}{2} \iint dx dx' \phi_a(x) G^{-1a}_b[\chi, A](x, x') \phi^b(x') \\ &\quad + \int dx \left\{ [\chi^2(x) - |A(x)|^2]/(2\lambda) - s(x)\chi(x) \right. \\ &\quad \left. + S^*(x)A(x) + S(x)A^*(x) + j^*(x)\phi(x) + j(x)\phi^*(x) \right\}, \end{aligned} \quad (3)$$

with

$$\begin{aligned} G^{-1a}_b[\chi, A] &= \{ G_0^{-1a}_b + V^a_b[\chi, A](x) \} \delta(x, x'), \\ G_0^{-1a}_b &= \begin{pmatrix} h_0 & 0 \\ 0 & h_0^* \end{pmatrix}, \quad h_0 = -\frac{\hbar^2 \nabla^2}{2m} - i\hbar \frac{\partial}{\partial t} - \mu, \\ V^a_b[\chi, A](x) &= \begin{pmatrix} \chi(x) \cosh \theta & -A(x) \sinh \theta \\ -A^*(x) \sinh \theta & \chi(x) \cosh \theta \end{pmatrix}. \end{aligned} \quad (4)$$

Here, we introduced a two-component notation with  $\phi^a(x) = \{ \phi(x), \phi^*(x) \}$  for  $a = 1, 2$ .  $\Phi(x)$  and  $J(x)$  signify the five-component fields and currents. The generating functional for connected graphs is

$$Z[J] = e^{iW[J]/\hbar} = \mathcal{N} \int D\Phi e^{iS[\Phi; J]/\hbar},$$

with  $S[\Phi; J]$  given by Eq. (3). Performing the path integral over the fields  $\phi_a$ , we obtain the effective action

$$\begin{aligned} \epsilon S_{\text{eff}}[\chi; J, \epsilon] &= \frac{1}{2} \iint dx dx' j_a(x) G[\chi]^a_b(x, x') j^a(x') \\ &\quad + \int dx \left\{ \frac{\chi_i(x) \chi^i(x)}{2\lambda} - S_i(x) \chi^i(x) - \frac{\hbar}{2i} \text{Tr} \ln [G^{-1}] \right\}, \end{aligned}$$

where  $\chi^i(x) = \{ \chi(x), A(x)/\sqrt{2}, A^*(x)/\sqrt{2} \}$ ,  $S^i(x) = \{ s(x), S(x)/\sqrt{2}, S^*(x)/\sqrt{2} \}$ . The small parameter  $\epsilon$  allows us to perform the remaining path integral over  $\chi^i$  using the stationary-phase approximation. As shown in Ref.20,  $\epsilon$  counts loops in the AF propagator in analogy with  $\hbar$ , and provides the loop expansion of the effective action in terms of  $\chi$  propagators. Next, we expand the effective action about the stationary points,  $\chi_0^i(x)$ , defined by  $\delta S_{\text{eff}}[\chi; j]/\delta \chi_i(x) = 0$ . Hence, we obtain

$$\begin{aligned} \frac{\chi_0(x)}{\lambda} &= \{ |\phi_0(x)|^2 + \frac{\hbar}{2i} \text{Tr}[G(x, x)] \} \cosh \theta + s(x), \\ \frac{A_0(x)}{\lambda} &= \{ \phi_0^2(x) + \frac{\hbar}{i} G^2_1(x, x) \} \sinh \theta + S(x), \end{aligned}$$

where we introduced the notations

$$\phi_0^a[\chi_0](x) = \int dx' G[\chi_0]^a_b(x, x') j^b(x').$$

We emphasize that both  $\chi_0$  and  $A_0$  include self-consistent fluctuations. Expanding the effective action about the stationary point, we write

$$\begin{aligned} S_{\text{eff}}[\chi; J] &= S_{\text{eff}}[\chi_0; J] + \frac{1}{2} \iint d^4x d^4x' D_{ij}^{-1}[\chi_0](x, x') \\ &\quad \times [\chi^i(x) - \chi_0^i(x)] [\chi^j(x') - \chi_0^j(x')] + \dots, \end{aligned} \quad (5)$$

where  $D_{ij}^{-1}(x, x')$  is given by the second-order derivatives,

$$D_{ij}^{-1}[\chi_0](x, x') = \left. \frac{\delta^2 S_{\text{eff}}[\chi^a]}{\delta \chi^i(x) \delta \chi^j(x')} \right|_{\chi_0},$$

evaluated at the stationary points. By keeping the gaussian fluctuations and Legendre transforming, the one-particle irreducible (1-PI) graphs generating functional

$$\begin{aligned} \Gamma[\Phi] &= \int dx j_\alpha(x) \phi^\alpha(x) - W[J] \\ &= \frac{1}{2} \iint dx dx' \phi_a(x) G^{-1}[\chi]^a_b(x, x') \phi^b(x') \\ &\quad - \int dx \left\{ \frac{\chi_i(x) \chi^i(x)}{2\lambda} - \frac{\hbar}{2i} \text{Tr} \{ \ln [G^{-1}[\chi](x, x)] \} \right. \\ &\quad \left. - \frac{\hbar \epsilon}{2i} \text{Tr} \ln [D_{ii}^{-1}[\Phi](x, x)] \right\} + \dots, \end{aligned} \quad (6)$$

is the negative of the classical action plus self-consistent one-loop corrections in the  $\phi_a$  and  $\chi_i$  propagators.

To leading order in the AF loop expansion (LOAF), one sets  $\epsilon = 0$  in the right-hand-side of (6). The static part of the effective action per unit volume is

$$\begin{aligned} V_{\text{eff}}[\Phi] &= (\chi \cosh \theta - \mu) |\phi|^2 - \frac{1}{2} (A^* \phi^2 + A \phi^{*2}) \sinh \theta \\ &\quad - \frac{\chi^2 - |A|^2}{2\lambda} + \frac{\hbar}{2i} \text{Tr} \{ \ln [G^{-1}[\chi]] \}. \end{aligned} \quad (7)$$

Translating (7) to the imaginary time formalism, we find

$$\frac{\hbar}{2i} \text{Tr} \ln [G^{-1}[\chi]] = \int \frac{d^3k}{(2\pi)^3} \left\{ \frac{\omega_k}{2} + \frac{1}{\beta} \ln [1 - e^{-\beta \omega_k}] \right\},$$

where  $\omega_k^2 = (\epsilon_k + \chi \cosh \theta - \mu)^2 - |A|^2 \sinh^2 \theta$  and  $\epsilon_k = k^2/(2m)$ . At the minimum, we have

$$\left. \frac{\delta V_{\text{eff}}[\Phi]}{\delta \phi^*} \right|_{\phi_0} = (\chi \cosh \theta - \mu) \phi_0 - A \sinh \theta \phi_0^* = 0. \quad (8)$$

Using the  $U(1)$  gauge symmetry, we choose  $\phi_0$  to be real. Then,  $A$  is real and the dispersion,  $\omega_k^2 = \epsilon_k(\epsilon_k + 2A \sinh \theta)$ , represents the Goldstone theorem. Next, we set  $\sinh \theta = 1$ , such that  $\omega_k$  reduces to the Bogoliubov dispersion,  $\omega_k = \sqrt{\epsilon_k(\epsilon_k + 2\lambda \phi_0^2)}$ , in the limit of vanishing quantum fluctuations in the anomalous density. We

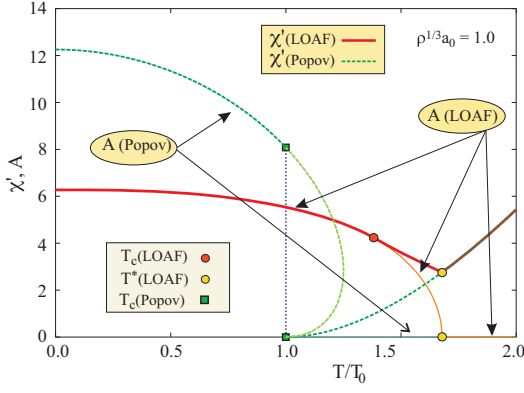


FIG. 1. (Color online) Normal density,  $\chi'$ , and anomalous density,  $A$ , from the LOAF and PA approximations, for  $\rho^{1/3}a_0 = 1$ .  $T_c$  and  $T^*$  indicate vanishing condensate density,  $\rho_0$ , and anomalous density,  $A$ , respectively. PA leads to a first-order phase transition, whereas LOAF predicts a second-order phase transition. We have that  $T_c = T^*$  in the PA, but not in LOAF. In LOAF  $\chi'$  and  $A$  are equal until  $T_c$ .

note that the leading-order (LO) in the large- $N$  expansion corresponds to  $\theta = 0$ . This leads to the noninteracting (NI) dispersion,  $\omega_k = \epsilon_k$ , and we conclude that the large- $N$  expansion is not a suitable starting point, because it is incompatible with the Bogoliubov spectrum.

Using standard regularization techniques[22], the renormalized effective potential is written as

$$V_{\text{eff}}[\Phi] = \chi'|\phi|^2 - \frac{1}{2}(A^*\phi^2 + A\phi^{*2}) - \frac{(\chi' + \mu)^2}{4\lambda} + \frac{|A|^2}{2\lambda} + \int \frac{d^3k}{(2\pi)^3} \left[ \frac{1}{2}(\omega_k - \epsilon_k - \chi' + \frac{|A|^2}{2\epsilon_k}) + \frac{1}{\beta} \ln(1 - e^{-\beta\omega_k}) \right],$$

where  $\chi' = \sqrt{2}\chi - \mu$  and  $\omega_k^2 = (\epsilon_k + \chi' + |A|)(\epsilon_k + \chi' - |A|)$ . The gap equations, obtained from  $\delta V_{\text{eff}}[\Phi]/\delta\chi^i = 0$ , are

$$\frac{A}{\lambda} = \phi^2 + A \int \frac{d^3k}{(2\pi)^3} \left\{ \frac{1 + 2n(\omega_k)}{2\omega_k} - \frac{1}{2\epsilon_k} \right\}, \quad (9)$$

$$\frac{\chi' + \mu}{2\lambda} = |\phi|^2 + \int \frac{d^3k}{(2\pi)^3} \left\{ \frac{\epsilon_k + \chi'}{2\omega_k} [1 + 2n(\omega_k)] - \frac{1}{2} \right\},$$

where  $n(\omega_k) = [\exp(\omega_k/k_B T) - 1]^{-1}$  is the Bose-Einstein particle distribution. At the minimum of the effective potential we have,  $(\chi'_0 - A_0)\phi_0 = 0$ , see Eq. (8), and we replace  $\mu$  by the physical density using  $\rho = -\partial V_{\text{eff}}[\Phi_0]/\partial\mu = (\chi'_0 + \mu)/(2\lambda)$ . The density is used to rescale Eqs. (9), and the ensuing phase diagram problem depends only on the dimensionless parameter,  $\rho^{1/3}a_0$ , and the coupling constant becomes  $\lambda = 8\pi\rho^{1/3}a_0$ . In the broken symmetry phase, we have  $\chi'_0 = A_0$  and the dispersion relation,  $\omega_k^2 = \epsilon_k(\epsilon_k + 2\chi'_0)$ . The condensate density is denoted by  $\rho_0 = \phi_0^2$ . At weak coupling and  $T = 0$ , our results coincide with the Bogoliubov (one-loop) approximation[14],  $\mu = 8\pi\rho a_0[1 + (32/3)\sqrt{\rho a_0^3/\pi}]$ .

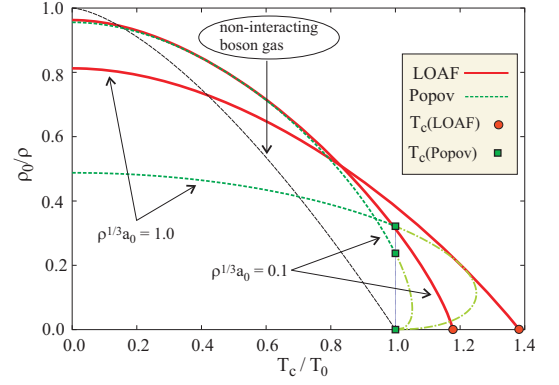


FIG. 2. (Color online) Temperature dependence of the condensate fractions from LOAF and PA, compared with the NI result, for  $\rho^{1/3}a_0 = 0.1$  and  $\rho^{1/3}a_0 = 1$ . Because at  $T_c$  the PA and NI dispersion relations are the same, PA does not change  $T_c$  relative to the NI case. LOAF increases  $T_c$ .

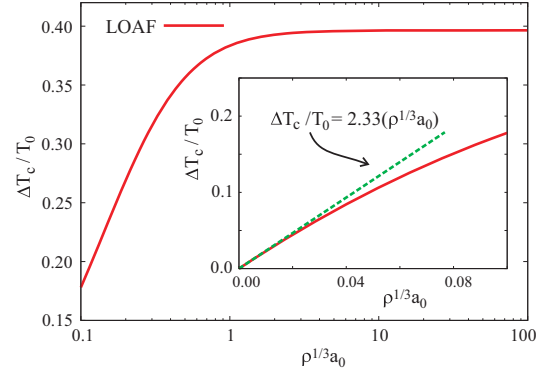


FIG. 3. (Color online) Relative change in  $T_c$  with respect to NI, as predicted by LOAF as a function of  $\rho^{1/3}a_0$ . The inset shows that in the weak-coupling regime, LOAF produces the same slope as the next-to-leading order large- $N$  expansion[17].

We compare the LOAF results with the predictions of the Popov bosonic approximation (PA)[23]. PA is generally recognized as an accurate theoretical description of experimental data in weakly-coupled dilute trapped Bose gases[24], as long as the densities of the condensed and noncondensed atoms are comparable with each other. Unfortunately, PA produces an artificial first-order phase transition at  $T_c$ . Formally, PA is obtained from Eq. (9) by setting  $A_0 = \chi'_0 = \lambda\rho_0$  and neglecting the quantum fluctuations in the anomalous density. With this substitution, the PA dispersion relation reads  $\omega_k^2 = \epsilon_k(\epsilon_k + 2\lambda\rho_0)$ .

In Fig. 1 we depict the temperature dependence of the normal density  $\chi'$ , and anomalous density,  $A$ , at constant  $\rho^{1/3}a_0$ , as derived using the LOAF and PA approximations. For illustrative purposes, we set  $\rho^{1/3}a_0 = 1$  and the temperature is scaled by its NI critical value,  $T_0 = (2\pi\hbar^2/m)[\rho/\zeta(3/2)]^{2/3}$ , where  $\zeta(x)$  is the Riemann zeta function. We identify two special temperatures, at  $T_c$  where the condensate density vanishes, and at  $T^*$

where the anomalous density,  $A$ , vanishes. These temperatures are the same in the PA formalism, but they are different in LOAF. The existence of a temperature range,  $T_c < T < T^*$ , for which the anomalous density,  $A$ , is nonzero despite a zero condensate fraction,  $\phi$ , is a fundamental prediction of LOAF. In this temperature range, the dispersion relation is expected to depart from the quadratic form predicted by the Popov approximation for  $T > T_c$ . Above  $T_c$  the solution of the PA equations becomes multivalued, indicating that the system undergoes a first-order phase transition at  $T_c$ . In contrast, LOAF predicts a second-order transition.

The temperature dependence of the condensate fraction,  $\rho_0/\rho$ , is depicted in Fig. 2 for two constant values of the dimensionless parameter  $\rho^{1/3}a_0$ , together with the NI result,  $\rho_0/\rho = 1 - (T/T_0)^{3/2}$ . Again, we observe that LOAF exhibits the correct second-order BEC phase transition behavior. Moreover, PA does not change  $T_c$  relative to the NI case, because in the PA case we have  $T_c = T^*$  and the PA and NI dispersion relations are the same at  $T_c$ . The LOAF approximation predicts an increase of  $T_c$  compared with the NI case.

As illustrated in Fig. 2, the LOAF and PA predictions may differ greatly even for temperatures,  $T \ll T_c$ . These differences are enhanced by a strengthening of the interaction between particles in the Bose gas (a larger value of  $\rho^{1/3}a_0$  indicates stronger coupling). The leading-order AF formalism produces a more realistic set of observables away from the weak-coupling limit because of its non-perturbative character. In contrast, PA is appropriate only in the case of a weakly-interacting gas of bosons. The former is made explicit by studying the LOAF prediction for the relative change in  $T_c$  with respect to  $T_0$ , as a function of  $\rho^{1/3}a_0$ . The inset in Fig. 3 demonstrates that in the weak-coupling regime,  $\rho^{1/3}a_0 \ll 1$ , LOAF produces the same slope of the linear departure derived by Baym *et al.*[17] using the large- $N$  expansion, but at next-to-leading order. The LOAF corrections to the critical temperature are due to the inclusion of self-consistent fluctuations effects in the mean-field  $\chi'$  and  $A$  densities. A summary of  $\Delta T_c/T_0$  theoretical predictions is found in Ref.14. For  $\rho^{1/3}a_0 \gg 1$ , LOAF predicts that  $\Delta T_c/T_0 \rightarrow 0.396$  when the system approaches the unitarity limit. Despite that most current experiments probe only the  $\rho^{1/3}a_0 \ll 1$  regime, future experiments[11] may access the medium-to-strongly interacting regime, and verify this non-perturbative prediction.

One can systematically improve upon the LOAF approximation by calculating the 1-PI action order-by-order in  $\epsilon$ . The broken  $U(1)$  symmetry Ward identities guarantee Goldstone's theorem order by order in  $\epsilon$  [20]. For time-dependent problems, however, this expansion is secular[25], and a further resummation is required. The latter is performed using the two-particle irreducible (2-PI) formalism[26]. A practical implementation of this approach is the bare-vertex approximation (BVA)[27]. The

BVA is an energy-momentum and particle-number conserving truncation of the Schwinger-Dyson infinite hierarchy of equations obtained by ignoring the derivatives of the self-energy, similarly to the Migdal's theorem[28] approach in condensed matter physics. The BVA proved effective in the case of classical and quantum  $\lambda\phi^4$  field theory problems[29] and can be applied to the BEC case.

To summarize, in this paper we introduce a new non-perturbative resummation formulation for the BEC problem. At mean-field level, this approach meets three important criteria for a satisfactory mean-field theory for weakly-interacting bosons[14]: i) the excitation spectrum is gapless (to preserve Goldstone's theorem), ii) LOAF reduces to the known results from Bogoliubov theory at  $T = 0$  and weak coupling, and iii) predicts a second-order BEC phase transition. The latter suggests that a AF formulation of the Lagrangian for systems of cold fermionic atoms may also impact the study of the BEC to BCS crossover in dilute fermionic atom systems[30].

Work performed in part under the auspices of the U.S. Department of Energy. The authors would like to thank E. Mottola and P.B. Littlewood for useful discussions.

- 
- [1] O. H. Kamerling, Proc. Roy. Acad. Amsterdam, **13**, 1903 (1911).
  - [2] P. L. Kapitza, Nature, **141**, 74 (1938); J. F. Allen and A. D. Misener, *ibid.*, **141**, 75 (1938).
  - [3] F. London, Nature, **141**, 643 (1938); Phys. Rev., **54**, 947 (1938).
  - [4] N. N. Bogoliubov, J. Phys. USSR, **11**, 23 (1947).
  - [5] L. D. Landau, J. Phys. USSR, **5**, 71 (1941).
  - [6] T. D. Lee, K. Huang, and C. N. Yang, Phys. Rev., **106**, 1135 (1957).
  - [7] Y. Shin, C. H. Schunck, A. Schirotzek, and W. Ketterle, Phys. Rev. Lett., **99**, 090403 (2007).
  - [8] P. O. Fedichev, M. W. Reynolds, and G. V. Shlyapnikov, Phys. Rev. Lett., **77**, 2921 (1996); B. D. Esry, C. H. Greene, and J. P. Burke, **83**, 1751 (1999).
  - [9] T. L. Ho, Phys. Rev. Lett., **92**, 090402 (2004); D. Blume, J. von Stecher, and C. H. Greene, **99**, 233201 (2007).
  - [10] A. J. Daley, J. M. Taylor, S. Diehl, M. Baranov, and P. Zoller, Phys. Rev. Lett., **102**, 040402 (2009).
  - [11] K. Henderson, H. Kelkar, T. C. Lee, B. Gutierrez-Medina, and M. G. Raizen, Europhys. Lett., **75**, 392 (2006); K. Henderson, C. Ryu, C. MacCormic, and M. Boshier, New J. Phys., **11**, 043030 (2009).
  - [12] P. C. Hohenberg and P. C. Martin, Ann. Phys., **34**, 291 (1965).
  - [13] A. Griffin, Phys. Rev. B, **53**, 9341 (1996).
  - [14] J. O. Andersen, Revs. Mod. Phys., **76**, 599 (2004).
  - [15] Collective modes have been measured in BECs, see J. M. Vogels, K. Xu, C. Raman, J. R. Abo-Shaeer, and W. Ketterle, Phys. Rev. Lett., **88**, 060402 (2002), using a method that provides an experimental verification of the fact that the  $\mathbf{q}$ -momentum quasi-particle is a superposition of  $\mathbf{q}$  and  $-\mathbf{q}$  waves. This mixing involves the anomalous density, so that the presence of an anomalous density

- above the BEC  $T_c$ , as predicted by our theory, may be tested not only by measuring the frequency dispersion, but also by testing the mixing.
- [16] In the Thomas-Fermi approximation, the local BEC density,  $\rho(x)$ , in a trapping potential,  $V_T(x)$ , follows from the density-dependent chemical potential,  $\mu(\rho) = \mu - V_T(x)$ . Taking the gradient of both sides, we find that the local trap force experienced by the bosons  $F_T(x) = -\nabla V_T(x)$  and the boson density gradient,  $\nabla\rho(x)$ , are proportional with a constant of proportionality equal to  $\partial\mu/\partial\rho$ , related to the compressibility,  $\nabla\rho(x)/F_T(x) = \delta\mu/\delta\rho$ . With the sensitive density profile measurement developed for fermion thermometry, experimentalists could, in principle, verify the compressibility calculation.
  - [17] G. Baym, J.-P. Blaizot, and J. Zinn-Justin, *Europhys. Lett.*, **49**, 150 (2000).
  - [18] S. Floerchinger and C. Wetterich, *Phys. Rev. A*, **77**, 053603 (2008).
  - [19] J. Hubbard, *Phys. Rev. Lett.*, **3**, 77 (1959); R. L. Stratonovich, *Doklady*, **2**, 416 (1958).
  - [20] C. Bender, F. Cooper, and G. Guralnik, *Ann. Phys.*, **109**, 165 (1977).
  - [21] S. Coleman, R. Jackiw, and H. D. Politzer, *Phys. Rev. D*, **10**, 2491 (1974); R. Root, *Phys. Rev. D*, **10**, 3322 (1974).
  - [22] T. Papenbrock and G. F. Bertsch, *Phys. Rev. C*, **59**, 2052 (1999).
  - [23] V. N. Popov, *Functional integrals and collective excitations* (Cambridge University Press, Cambridge, England, 1987).
  - [24] F. Dalfovo, S. Giorgini, L. P. Pitaevskii, and S. Stringari, *Rev. Mod. Phys.*, **71**, 463 (1999).
  - [25] B. Mihaila, J. F. Dawson, and F. Cooper, *Phys. Rev. D*, **63**, 096003 (2001).
  - [26] G. Baym, *Phys. Rev.*, **127**, 1391 (1962); J. M. Cornwall, R. Jackiw, and E. Tomboulis, *Phys. Rev. D*, **10**, 2428 (1974).
  - [27] K. B. Blagoev, F. Cooper, J. F. Dawson, and B. Mihaila, *Phys. Rev. D*, **64**, 125003 (2001).
  - [28] A. B. Migdal, *Sov. Phys. JETP*, **7**, 996 (1958).
  - [29] F. Cooper, J. F. Dawson, and B. Mihaila, *Phys. Rev. D*, **67**, 051901R (2003); **67**, 056003 (2003); B. Mihaila, **68**, 036002 (2003).
  - [30] K. Levin, Q. J. Chen, C. C. Chien, and Y. He, *Ann. Phys.*, **325**, 233 (2010).